

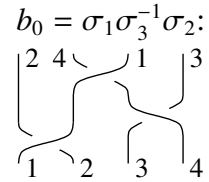
# A NOTE ON THE UNITARITY PROPERTY OF THE GASSNER INVARIANT

DROR BAR-NATAN

**ABSTRACT.** We give a 3-page description of the Gassner invariant (or representation) of braids (or pure braids), along with a description and a proof of its unitarity property.

The unitarity of the Gassner representation [Ga] of the pure braid group was discussed by many authors (e.g. [Lo, Ab, K LW]) and from several points of view, yet without exposing how utterly simple the formulas turn out to be<sup>1</sup>. When the present author needed quick and easy formulas, he couldn't find them. This note is written in order to rectify this situation (but with no discussion of theory). I was heavily influenced by a similar discussion of the unitarity of the Burau representation in [KT, Section 3.1.2].

Let  $n$  be a natural number. The braid group  $B_n$  on  $n$  strands is the group with generators  $\sigma_i$ , for  $1 \leq i \leq n-1$ , and with relations  $\sigma_i \sigma_j = \sigma_j \sigma_i$  when  $|i-j| > 1$  and  $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$  when  $1 \leq i \leq n-2$ . A standard way to depict braids, namely elements of  $B_n$ , appears on the right. Braids are made of strands that are indexed 1 through  $n$  at the bottom. The generator  $\sigma_i$  denotes a positive crossing between the strand at position  $\#i$  as counted just below the horizontal level of that crossing, and the strand just to its right. Note that with the strands indexed at the bottom, the two strands participating in a crossing corresponding to  $\sigma_i$  may have arbitrary indices, depending on the permutation induced by the braids below the level of that crossing.



Let  $t$  be a formal variable and let  $U_i(t) = U_{n,i}(t)$  denote the  $n \times n$  identity matrix with its  $2 \times 2$  block at rows  $i$  and  $i+1$  and columns  $i$  and  $i+1$  replaced by  $\begin{pmatrix} 1-t & 1 \\ t & 0 \end{pmatrix}$ . Let  $U_i^{-1}(t)$  be the inverse of  $U_i(t)$ ; it is the  $n \times n$  identity matrix with the block at  $\{i, i+1\} \times \{i, i+1\}$  replaced by  $\begin{pmatrix} 0 & \bar{t} \\ 1 & 1-\bar{t} \end{pmatrix}$ , where  $\bar{t}$  denotes  $t^{-1}$ .

$$U_{5;3}(t) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1-t & 1 & 0 \\ 0 & 0 & t & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Let  $b$  be a braid  $b = \prod_{\alpha=1}^k \sigma_{i_\alpha}^{s_\alpha}$ , where the  $s_\alpha$  are signs and where products are taken from left to right. Let  $j_\alpha$  be the index of the “over” strand at crossing  $\# \alpha$  in  $b$ . The Gassner invariant  $\Gamma(b)$  of  $b$  is given by the formula on the right. It is a Laurent polynomial in  $n$  formal variables  $t_1, \dots, t_n$ , with coefficients in  $\mathbb{Z}$ .

$$\Gamma(b) := \prod_{\alpha=1}^k U_{i_\alpha}^{s_\alpha}(t_{j_\alpha})$$

*Date:* September 1, 2014; first edition: June 29, 2014.

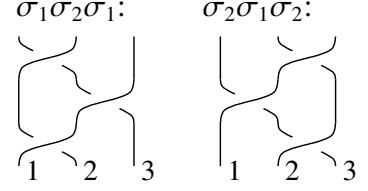
*2010 Mathematics Subject Classification.* 57M25.

*Key words and phrases.* Braids, Unitarity, Gassner, Burau.

This work was partially supported by NSERC grant RGPIN 262178. The full T<sub>E</sub>X sources are at <http://drorbn.net/AcademicPensieve/2014-06/UnitarityOfGassner/>. Updated less often: [arXiv:1406.7632](http://arxiv.org/abs/1406.7632).

<sup>1</sup>Partially this is because the formulas are simplest when extended a “Gassner invariant” defined on the full braid group, but then it is not a representation and it is not unitary. Yet it has an easy “unitarity property”; see below.

For example,  $\Gamma(\sigma_1\sigma_2\sigma_1) = U_1(t_1)U_2(t_1)U_1(t_2)$  while  $\Gamma(\sigma_2\sigma_1\sigma_2) = U_2(t_2)U_1(t_1)U_2(t_1)$ . The equality of these two matrix products constitutes the bulk of the proof of the well-definedness of  $\Gamma$ , and the rest is even easier. The verification of this equality is a routine exercise in  $3 \times 3$  matrix multiplication. Impatient readers may find it in the *Mathematica* notebook that accompanies this note, [BN].



A second example is the braid  $b_0$  of the first figure. Here and in [BN],

$$\Gamma(b_0) = U_1(t_1)U_3^{-1}(t_4)U_2(t_1) = \begin{pmatrix} 1-t_1 & 1-t_1 & 1 & 0 \\ t_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \bar{t}_4 \\ 0 & t_1 & 0 & 1-\bar{t}_4 \end{pmatrix}$$

Given a permutation  $\tau = [\tau 1, \dots, \tau n]$  of  $1, \dots, n$ , let  $\Omega(\tau)$  be the triangular  $n \times n$  matrix shown on the right (diagonal entries  $(1-t_{\tau i})^{-1}$ , 1's below the diagonal, 0's above). Let  $\iota$  denote the identity permutation  $[1, 2, \dots, n]$ .

$$\Omega(\tau) := \begin{pmatrix} (1-t_{\tau 1})^{-1} & 0 & \dots & 0 \\ 1 & (1-t_{\tau 2})^{-1} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \dots & (1-t_{\tau n})^{-1} \end{pmatrix}$$

**Theorem.** Let  $b$  be a braid that induces a strand permutation  $\tau = [\tau 1, \dots, \tau n]$  (meaning, the strand indices that appear at the top of  $b$  are  $\tau 1, \tau 2, \dots, \tau n$ ). Let  $\gamma = \Gamma(b)$  be the Gassner invariant of  $b$ . Then  $\gamma$  satisfies the “unitarity property”

$$(1) \quad \Omega(\tau)\gamma^{-1} = \bar{\gamma}^T \Omega(\iota), \quad \text{or equivalently,} \quad \gamma^{-1} = \Omega(\tau)^{-1} \bar{\gamma}^T \Omega(\iota),$$

where  $\bar{\gamma}$  is  $\gamma$  subject to the substitution  $\forall i \ t_i \rightarrow \bar{t}_i := t_i^{-1}$ , and  $\bar{\gamma}^T$  is the transpose matrix of  $\bar{\gamma}$ .

*Proof.* A direct and simple-minded computation proves Equation (1) for  $b = \sigma_i$  and for  $b = \sigma_i^{-1}$ , namely for  $\gamma = U_i(t_i)$  and for  $\gamma = U_i^{-1}(t_{i+1})$  (impatient readers see [BN]), and then, clearly, using the second form of Equation (1), the statement generalizes to products with all the intermediate  $\Omega(\tau)^{-1}\Omega(\tau)$  pairs cancelling out nicely.  $\square$

If the Gassner invariant  $\Gamma$  is restricted to pure braids, namely to braids that induce the identity permutation, it becomes multiplicative and then it can be called “the Gassner representation” (in general  $\Gamma$  can be recast as a homomorphism into  $M_{n \times n}(\mathbb{Z}[t_i, \bar{t}_i]) \rtimes S_n$ , where  $S_n$  acts on matrices by permuting the variables  $t_i$  appearing in their entries).

For pure braids  $\Omega(\tau) = \Omega(\iota) =: \Omega$  and hence by conjugating (in the  $t_i \rightarrow 1/t_i$  sense) and transposing Equation (1) and replacing  $\gamma$  by  $\gamma^{-1}$ , we find that the theorem also holds if  $\Omega$  is replaced by  $\bar{\Omega}^T$ . Hence, extending the coefficients to  $\mathbb{C}$ , the theorem also holds if  $\Omega$  is replaced by  $\Psi := i\Omega - i\bar{\Omega}^T$ , which is formally Hermitian ( $\bar{\Psi}^T = \Psi$ ).

If the  $t_i$ 's are specialized to complex numbers of unit norm then inversion is the same as complex conjugation. If also the  $t_i$ 's are sufficiently close to 1 and have positive imaginary parts, then  $\Psi$  is dominated by its main diagonal entries, which are real, positive, and large, and hence  $\Psi$  is positive definite and genuinely Hermitian. Thus in that case, the Gassner representation is unitary in the standard sense of the word, relative to the inner product on  $\mathbb{C}^n$  defined by  $\Psi$ .

We remark is that the Gassner representation easily extends to a representation of pure v/w-braids. See e.g. [BND, Sections 2.1.2 and 2.2], where the generators  $\sigma_{ij}$  are described (they are *not* generators of the ordinary pure braid group). Simply set  $\Gamma(\sigma_{ij})^{\pm 1} = U_{ij}^{\pm 1}$  where  $U_{ij}$  is the  $n \times n$  identity matrix with its  $2 \times 2$  block at rows  $i$  and  $j$  and columns  $i$  and  $j$  replaced by  $\begin{pmatrix} 1 & 1-t_i \\ 0 & t_i \end{pmatrix}$ . Yet

on v/w-braids  $\Gamma$  does not satisfy the unitarity property of this note and I'd be very surprised if it is at all unitary.

We also remark that there is an alternative form  $\Gamma'$  for the Gassner representation of pure v/w-braids, defined by  $\Gamma'(\sigma_{ij})^{\pm 1} = V_{ij}^{\pm 1}$  where  $V_{ij}$  is the  $n \times n$  identity matrix with its  $2 \times 2$  block at rows  $i$  and  $j$  and columns  $i$  and  $j$  replaced by  $\begin{pmatrix} 1 & 1 - t_j \\ 0 & t_i \end{pmatrix}$ . Clearly,  $U_{ij}$  and  $V_{ij}$  are conjugate;  $V_{ij} = D^{-1}U_{ij}D$ , with  $D$  the diagonal matrix whose  $(i, i)$  entry is  $1 - t_i$  for every  $i$ . Hence on ordinary pure braids and for appropriate values of the  $t_i$ 's (as above),  $\Gamma'$  is also unitary, relative to the Hermitian inner product defined by the matrix

$$\Psi' := \bar{D}^T \Psi D = i \bar{D}^T (\Omega - \bar{\Omega}^T) D$$

whose printed form is better avoided (yet it appears at the end of [BN]).

#### REFERENCES

- [Ab] M. N. Abdulrahim, *A Faithfulness Criterion for the Gassner Representation of the Pure Braid Group*, Proceedings of the American Mathematical Society **125-5** (1997) 1249–1257.
- [BN] D. Bar-Natan, *UnitarityOfGassnerDemo.nb*, a *Mathematica* notebook at <http://drorbn.net/AcademicPensieve/2014-06/UnitarityOfGassner/>.
- [BND] D. Bar-Natan and Z. Dancso, *Finite Type Invariants of w-Knotted Objects I: w-Knots and the Alexander Polynomial*, <http://drorbn.net/AcademicPensieve/Projects/WK01/> and [arXiv:1405.1956](https://arxiv.org/abs/1405.1956).
- [Ga] B. J. Gassner, *On Braid Groups*, Ph.D. thesis, New York University, 1959.
- [KT] C. Kassel and V. Turaev, *Braid Groups*, Springer GTM **247**, 2008.
- [KLW] P. Kirk, C. Livingston, and Z. Wang, *The Gassner Representation for String Links*, Communications in Contemporary Mathematics **3-1** (2001) 87–136, [arXiv:math/9806035](https://arxiv.org/abs/math/9806035).
- [Lo] D. D. Long, *On the Linear Representation of Braid Groups*, Transactions of the American Mathematical Society **311-2** (1989) 535–560.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF TORONTO, TORONTO ONTARIO M5S 2E4, CANADA  
*E-mail address*: [drorbn@math.toronto.edu](mailto:drorbn@math.toronto.edu)  
*URL*: <http://www.math.toronto.edu/~drorbn>

Pensieve header: Mathematica notebook accompanying "A Note on the Unitarity Property of the Gassner Invariant" by Dror Bar-Natan, <http://drorbn.net/AcademicPensieve/2014-06/UnitarityOf-Gassner/>.

## Definitions.

```

Ui[_] := ReplacePart[IdentityMatrix[n],
  {{i, i} → 1 - t, {i, i + 1} → 1,
   {i + 1, i} → t, {i + 1, i + 1} → 0}];
Ui-[_] := Inverse[Ui[t]];
Ω[ε---] := Table[
  Which[i < j, 0, i == j, (1 - t{ε}[[i]])-1, i > j, 1],
  {i, n}, {j, n}];
 $\overline{X}$  := X /. ti → 1 / ti;
Ui,j[_] := ReplacePart[IdentityMatrix[n],
  {{i, i} → 1, {i, j} → 1 - ti,
   {j, i} → 0, {j, j} → ti}}];
Vi,j[_] := ReplacePart[IdentityMatrix[n],
  {{i, i} → 1, {i, j} → 1 - tj,
   {j, i} → 0, {j, j} → tj}}];
DD := DiagonalMatrix[Table[1 - ti, {i, n}]];

```

## The named matrices.

**n = 5; MatrixForm /@ Simplify /@ {U<sub>3</sub>[t], U<sub>3</sub><sup>-</sup>[t]}**

$$\left\{ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1-t & 1 & 0 \\ 0 & 0 & t & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{t} & 0 \\ 0 & 0 & 1 & \frac{-1+t}{t} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \right\}$$

**n = 3; MatrixForm /@ Simplify /@ {Ω[2, 3, 1], Inverse[Ω[2, 3, 1]]}**

$$\left\{ \begin{pmatrix} \frac{1}{1-t_2} & 0 & 0 \\ 1 & \frac{1}{1-t_3} & 0 \\ 1 & 1 & \frac{1}{1-t_1} \end{pmatrix}, \begin{pmatrix} 1-t_2 & 0 & 0 \\ -(-1+t_2)(-1+t_3) & 1-t_3 & 0 \\ -(-1+t_1)(-1+t_2)t_3 & -(-1+t_1)(-1+t_3) & 1-t_1 \end{pmatrix} \right\}$$

**n = 5; MatrixForm /@ {U<sub>4,1</sub>, V<sub>4,1</sub>, DD}**

$$\left\{ \begin{pmatrix} t_4 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1-t_4 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} t_4 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1-t_1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1-t_1 & 0 & 0 & 0 & 0 \\ 0 & 1-t_2 & 0 & 0 & 0 \\ 0 & 0 & 1-t_3 & 0 & 0 \\ 0 & 0 & 0 & 1-t_4 & 0 \\ 0 & 0 & 0 & 0 & 1-t_5 \end{pmatrix} \right\}$$

The R3 move

```
n = 3; MatrixForm /@ Simplify /@ {U1[t1].U2[t1].U1[t2], U2[t2].U1[t1].U2[t1]}
```

$$\left\{ \begin{pmatrix} 1-t_1 & 1-t_1 & 1 \\ -t_1(-1+t_2) & t_1 & 0 \\ t_1 t_2 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1-t_1 & 1-t_1 & 1 \\ -t_1(-1+t_2) & t_1 & 0 \\ t_1 t_2 & 0 & 0 \end{pmatrix} \right\}$$

The unitarity property for the generators.

```
n = 5; γ = U3[t3];
```

```
MatrixForm /@
```

```
Simplify /@ {Ω[1, 2, 4, 3, 5].Inverse[γ], Transpose[γ̄].Ω[1, 2, 3, 4, 5]}
```

$$\left\{ \begin{pmatrix} \frac{1}{1-t_1} & 0 & 0 & 0 & 0 \\ 1 & \frac{1}{1-t_2} & 0 & 0 & 0 \\ 1 & 1 & 0 & \frac{1}{t_3-t_3 t_4} & 0 \\ 1 & 1 & \frac{1}{1-t_3} & 0 & 0 \\ 1 & 1 & 1 & 1 & \frac{1}{1-t_5} \end{pmatrix}, \begin{pmatrix} \frac{1}{1-t_1} & 0 & 0 & 0 & 0 \\ 1 & \frac{1}{1-t_2} & 0 & 0 & 0 \\ 1 & 1 & 0 & \frac{1}{t_3-t_3 t_4} & 0 \\ 1 & 1 & \frac{1}{1-t_3} & 0 & 0 \\ 1 & 1 & 1 & 1 & \frac{1}{1-t_5} \end{pmatrix} \right\}$$

```
n = 5; γ = U3̄[t4];
```

```
MatrixForm /@
```

```
FullSimplify /@ {Ω[1, 2, 4, 3, 5].Inverse[γ], Transpose[γ̄].Ω[1, 2, 3, 4, 5]}
```

$$\left\{ \begin{pmatrix} \frac{1}{1-t_1} & 0 & 0 & 0 & 0 \\ 1 & \frac{1}{1-t_2} & 0 & 0 & 0 \\ 1 & 1 & 1 & \frac{1}{1-t_4} & 0 \\ 1 & 1 & 1 - \frac{t_3 t_4}{-1+t_3} & 1 & 0 \\ 1 & 1 & 1 & 1 & \frac{1}{1-t_5} \end{pmatrix}, \begin{pmatrix} \frac{1}{1-t_1} & 0 & 0 & 0 & 0 \\ 1 & \frac{1}{1-t_2} & 0 & 0 & 0 \\ 1 & 1 & 1 & \frac{1}{1-t_4} & 0 \\ 1 & 1 & 1 - \frac{t_3 t_4}{-1+t_3} & 1 & 0 \\ 1 & 1 & 1 & 1 & \frac{1}{1-t_5} \end{pmatrix} \right\}$$

The braid  $b_0 = \sigma_1 \sigma_3^{-1} \sigma_2$ :

$n = 4$ ; `MatrixForm`[ $\gamma_0 = U_1[t_1] \cdot U_3[t_4] \cdot U_2[t_1]$ ]

$$\begin{pmatrix} 1-t_1 & 1-t_1 & 1 & 0 \\ t_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{t_4} \\ 0 & t_1 & 0 & -\frac{1-t_4}{t_4} \end{pmatrix}$$

The unitarity property for  $b_0$ .

`MatrixForm` /@ `Simplify` /@ { $\Omega[2, 4, 1, 3] \cdot \text{Inverse}[\gamma_0]$ ,  $\text{Transpose}[\overline{\gamma_0}] \cdot \Omega[1, 2, 3, 4]$ }

$$\left\{ \begin{pmatrix} 0 & \frac{1}{t_1-t_1 t_2} & 0 & 0 \\ 0 & \frac{1}{t_1} & \frac{1}{t_1} & \frac{1}{t_1-t_1 t_4} \\ \frac{1}{1-t_1} & 0 & 0 & 0 \\ 1 & 1 & -\frac{1+t_3(-1+t_4)}{-1+t_3} & 1 \end{pmatrix}, \begin{pmatrix} 0 & \frac{1}{t_1-t_1 t_2} & 0 & 0 \\ 0 & \frac{1}{t_1} & \frac{1}{t_1} & \frac{1}{t_1-t_1 t_4} \\ \frac{1}{1-t_1} & 0 & 0 & 0 \\ 1 & 1 & -\frac{1+t_3(-1+t_4)}{-1+t_3} & 1 \end{pmatrix} \right\}$$

On to w-braids

$n = 3$ ; `MatrixForm` /@ `Simplify` /@ { $U_{1,2} \cdot U_{1,3} \cdot U_{2,3}$ ,  $U_{2,3} \cdot U_{1,3} \cdot U_{1,2}$ }

$$\left\{ \begin{pmatrix} 1 & 1-t_1 & 1-t_1 \\ 0 & t_1 & -t_1(-1+t_2) \\ 0 & 0 & t_1 t_2 \end{pmatrix}, \begin{pmatrix} 1 & 1-t_1 & 1-t_1 \\ 0 & t_1 & -t_1(-1+t_2) \\ 0 & 0 & t_1 t_2 \end{pmatrix} \right\}$$

$n = 3$ ; `MatrixForm` /@ `Simplify` /@ { $U_{1,2} \cdot U_{1,3}$ ,  $U_{1,3} \cdot U_{1,2}$ }

$$\left\{ \begin{pmatrix} 1 & 1-t_1 & 1-t_1 \\ 0 & t_1 & 0 \\ 0 & 0 & t_1 \end{pmatrix}, \begin{pmatrix} 1 & 1-t_1 & 1-t_1 \\ 0 & t_1 & 0 \\ 0 & 0 & t_1 \end{pmatrix} \right\}$$

The “Other Gassner”  $\Gamma'$

$n = 4$ ; `MatrixForm` /@ `Simplify` /@ { $V_{4,1}$ ,  $\text{Inverse}[DD] \cdot U_{4,1} \cdot DD$ }

$$\left\{ \begin{pmatrix} t_4 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1-t_1 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} t_4 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1-t_1 & 0 & 0 & 1 \end{pmatrix} \right\}$$

```

n = 4; MatrixForm /@ Simplify /@ {
  Transpose[DD].Ω[1, 2, 3, 4].DD,
  Ψ' = i Transpose[DD].(Ω[1, 2, 3, 4] - Transpose[Ω[1, 2, 3, 4]]) . DD
}

```

$$\left\{ \begin{pmatrix} 1 - \frac{1}{t_1} & 0 & 0 & 0 \\ (1 - t_1) \left(1 - \frac{1}{t_2}\right) & 1 - \frac{1}{t_2} & 0 & 0 \\ (1 - t_1) \left(1 - \frac{1}{t_3}\right) & (1 - t_2) \left(1 - \frac{1}{t_3}\right) & 1 - \frac{1}{t_3} & 0 \\ (1 - t_1) \left(1 - \frac{1}{t_4}\right) & (1 - t_2) \left(1 - \frac{1}{t_4}\right) & (1 - t_3) \left(1 - \frac{1}{t_4}\right) & 1 - \frac{1}{t_4} \end{pmatrix}, \right.$$

$$\left\{ \begin{pmatrix} \frac{i(-1+t_1^2)}{t_1} & i\left(-1+\frac{1}{t_1}\right)(1-t_2) & i\left(-1+\frac{1}{t_1}\right)(1-t_3) & i\left(-1+\frac{1}{t_1}\right)(1-t_4) \\ -\frac{i(-1+t_1)(-1+t_2)}{t_2} & \frac{i(-1+t_2^2)}{t_2} & i\left(-1+\frac{1}{t_2}\right)(1-t_3) & i\left(-1+\frac{1}{t_2}\right)(1-t_4) \\ -\frac{i(-1+t_1)(-1+t_3)}{t_3} & -\frac{i(-1+t_2)(-1+t_3)}{t_3} & \frac{i(-1+t_3^2)}{t_3} & i\left(-1+\frac{1}{t_3}\right)(1-t_4) \\ -\frac{i(-1+t_1)(-1+t_4)}{t_4} & -\frac{i(-1+t_2)(-1+t_4)}{t_4} & -\frac{i(-1+t_3)(-1+t_4)}{t_4} & \frac{i(-1+t_4^2)}{t_4} \end{pmatrix} \right\}$$